

Section A: Pure Mathematics

- 1 The points S , T , U and V have coordinates (s, ms) , (t, mt) , (u, nu) and (v, nv) , respectively. The lines SV and UT meet the line $y = 0$ at the points with coordinates $(p, 0)$ and $(q, 0)$, respectively. Show that

$$p = \frac{(m-n)sv}{ms-nv},$$

and write down a similar expression for q .

Given that S and T lie on the circle $x^2 + (y - c)^2 = r^2$, find a quadratic equation satisfied by s and by t , and hence determine st and $s + t$ in terms of m , c and r .

Given that S , T , U and V lie on the above circle, show that $p + q = 0$.

- 2 (i) Let $y = \sum_{n=0}^{\infty} a_n x^n$, where the coefficients a_n are independent of x and are such that this series and all others in this question converge. Show that

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

and write down a similar expression for y'' .

Write out explicitly each of the three series as far as the term containing a_3 .

- (ii) It is given that y satisfies the differential equation

$$xy'' - y' + 4x^3y = 0.$$

By substituting the series of part (i) into the differential equation and comparing coefficients, show that $a_1 = 0$.

Show that, for $n \geq 4$,

$$a_n = -\frac{4}{n(n-2)} a_{n-4},$$

and that, if $a_0 = 1$ and $a_2 = 0$, then $y = \cos(x^2)$.

Find the corresponding result when $a_0 = 0$ and $a_2 = 1$.

3 The function $f(t)$ is defined, for $t \neq 0$, by

$$f(t) = \frac{t}{e^t - 1}.$$

- (i) By expanding e^t , show that $\lim_{t \rightarrow 0} f(t) = 1$. Find $f'(t)$ and evaluate $\lim_{t \rightarrow 0} f'(t)$.
- (ii) Show that $f(t) + \frac{1}{2}t$ is an even function. [**Note:** A function $g(t)$ is said to be *even* if $g(t) \equiv g(-t)$.]
- (iii) Show with the aid of a sketch that $e^t(1-t) \leq 1$ and deduce that $f'(t) \neq 0$ for $t \neq 0$.

Sketch the graph of $f(t)$.

4 For any given (suitable) function f , the *Laplace transform* of f is the function F defined by

$$F(s) = \int_0^{\infty} e^{-st}f(t)dt \quad (s > 0).$$

- (i) Show that the Laplace transform of $e^{-bt}f(t)$, where $b > 0$, is $F(s+b)$.
- (ii) Show that the Laplace transform of $f(at)$, where $a > 0$, is $a^{-1}F(\frac{s}{a})$.
- (iii) Show that the Laplace transform of $f'(t)$ is $sF(s) - f(0)$.
- (iv) In the case $f(t) = \sin t$, show that $F(s) = \frac{1}{s^2 + 1}$.

Using only these four results, find the Laplace transform of $e^{-pt} \cos qt$, where $p > 0$ and $q > 0$.

5 The numbers x , y and z satisfy

$$\begin{aligned}x + y + z &= 1 \\x^2 + y^2 + z^2 &= 2 \\x^3 + y^3 + z^3 &= 3.\end{aligned}$$

Show that

$$yz + zx + xy = -\frac{1}{2}.$$

Show also that $x^2y + x^2z + y^2z + y^2x + z^2x + z^2y = -1$, and hence that

$$xyz = \frac{1}{6}.$$

Let $S_n = x^n + y^n + z^n$. Use the above results to find numbers a , b and c such that the relation

$$S_{n+1} = aS_n + bS_{n-1} + cS_{n-2},$$

holds for all n .

6 Show that $|e^{i\beta} - e^{i\alpha}| = 2 \sin \frac{1}{2}(\beta - \alpha)$ for $0 < \alpha < \beta < 2\pi$. Hence show that

$$|e^{i\alpha} - e^{i\beta}| |e^{i\gamma} - e^{i\delta}| + |e^{i\beta} - e^{i\gamma}| |e^{i\alpha} - e^{i\delta}| = |e^{i\alpha} - e^{i\gamma}| |e^{i\beta} - e^{i\delta}|,$$

where $0 < \alpha < \beta < \gamma < \delta < 2\pi$.

Interpret this result as a theorem about cyclic quadrilaterals.

- 7 (i) The functions $f_n(x)$ are defined for $n = 0, 1, 2, \dots$, by

$$f_0(x) = \frac{1}{1+x^2} \quad \text{and} \quad f_{n+1}(x) = \frac{df_n(x)}{dx}.$$

Prove, for $n \geq 1$, that

$$(1+x^2)f_{n+1}(x) + 2(n+1)xf_n(x) + n(n+1)f_{n-1}(x) = 0.$$

- (ii) The functions $P_n(x)$ are defined for $n = 0, 1, 2, \dots$, by

$$P_n(x) = (1+x^2)^{n+1}f_n(x).$$

Find expressions for $P_0(x)$, $P_1(x)$ and $P_2(x)$.

Prove, for $n \geq 0$, that

$$P_{n+1}(x) - (1+x^2)\frac{dP_n(x)}{dx} + 2(n+1)xP_n(x) = 0,$$

and that $P_n(x)$ is a polynomial of degree n .

- 8 Let m be a positive integer and let n be a non-negative integer.

- (i) Use the result $\lim_{t \rightarrow \infty} e^{-mt}t^n = 0$ to show that

$$\lim_{x \rightarrow 0} x^m(\ln x)^n = 0.$$

By writing x^x as $e^{x \ln x}$ show that

$$\lim_{x \rightarrow 0} x^x = 1.$$

- (ii) Let $I_n = \int_0^1 x^m(\ln x)^n dx$. Show that

$$I_{n+1} = -\frac{n+1}{m+1}I_n$$

and hence evaluate I_n .

- (iii) Show that

$$\int_0^1 x^x dx = 1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 - \left(\frac{1}{4}\right)^4 + \dots.$$

Section B: Mechanics

9 A particle is projected under gravity from a point P and passes through a point Q . The angles of the trajectory with the positive horizontal direction at P and at Q are θ and ϕ , respectively. The angle of elevation of Q from P is α .

(i) Show that $\tan \theta + \tan \phi = 2 \tan \alpha$.

(ii) It is given that there is a second trajectory from P to Q with the same speed of projection. The angles of this trajectory with the positive horizontal direction at P and at Q are θ' and ϕ' , respectively. By considering a quadratic equation satisfied by $\tan \theta$, show that $\tan(\theta + \theta') = -\cot \alpha$. Show also that $\theta + \theta' = \pi + \phi + \phi'$.

10 A light spring is fixed at its lower end and its axis is vertical. When a certain particle P rests on the top of the spring, the compression is d . When, instead, P is dropped onto the top of the spring from a height h above it, the compression at time t after P hits the top of the spring is x . Obtain a second-order differential equation relating x and t for $0 \leq t \leq T$, where T is the time at which P first loses contact with the spring.

Find the solution of this equation in the form

$$x = A + B \cos(\omega t) + C \sin(\omega t),$$

where the constants A , B , C and ω are to be given in terms of d , g and h as appropriate.

Show that

$$T = \sqrt{d/g} \left(2\pi - 2 \arctan \sqrt{2h/d} \right).$$

11 A comet in deep space picks up mass as it travels through a large stationary dust cloud. It is subject to a gravitational force of magnitude Mf acting in the direction of its motion. When it entered the cloud, the comet had mass M and speed V . After a time t , it has travelled a distance x through the cloud, its mass is $M(1 + bx)$, where b is a positive constant, and its speed is v .

(i) In the case when $f = 0$, write down an equation relating V , x , v and b . Hence find an expression for x in terms of b , V and t .

(ii) In the case when f is a non-zero constant, use Newton's second law in the form

$$\text{force} = \text{rate of change of momentum}$$

to show that

$$v = \frac{ft + V}{1 + bx}.$$

Hence find an expression for x in terms of b , V , f and t .

Show that it is possible, if b , V and f are suitably chosen, for the comet to move with constant speed. Show also that, if the comet does not move with constant speed, its speed tends to a constant as $t \rightarrow \infty$.

Section C: Probability and Statistics

- 12 (i)** Albert tosses a fair coin k times, where k is a given positive integer. The number of heads he gets is X_1 . He then tosses the coin X_1 times, getting X_2 heads. He then tosses the coin X_2 times, getting X_3 heads. The random variables X_4, X_5, \dots are defined similarly. Write down $E(X_1)$.

By considering $E(X_2 \mid X_1 = x_1)$, or otherwise, show that $E(X_2) = \frac{1}{4}k$.

Find $\sum_{i=1}^{\infty} E(X_i)$.

- (ii)** Bertha has k fair coins. She tosses the first coin until she gets a tail. The number of heads she gets before the first tail is Y_1 . She then tosses the second coin until she gets a tail and the number of heads she gets with this coin before the first tail is Y_2 . The random variables Y_3, Y_4, \dots, Y_k are defined similarly, and $Y = \sum_{i=1}^k Y_i$.

Obtain the probability generating function of Y , and use it to find $E(Y)$, $\text{Var}(Y)$ and $P(Y = r)$.

- 13 (i)** The point P lies on the circumference of a circle of unit radius and centre O . The angle, θ , between OP and the positive x -axis is a random variable, uniformly distributed on the interval $0 \leq \theta < 2\pi$. The cartesian coordinates of P with respect to O are (X, Y) . Find the probability density function for X , and calculate $\text{Var}(X)$.

Show that X and Y are uncorrelated and discuss briefly whether they are independent.

- (ii)** The points P_i ($i = 1, 2, \dots, n$) are chosen independently on the circumference of the circle, as in part (i), and have cartesian coordinates (X_i, Y_i) . The point \bar{P} has coordinates (\bar{X}, \bar{Y}) , where $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ and $\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$. Show that \bar{X} and \bar{Y} are uncorrelated.

Show that, for large n , $P\left(|\bar{X}| \leq \sqrt{\frac{2}{n}}\right) \approx 0.95$.